4.3 Conics

What you should learn

- Recognize the four basic conics: circles, ellipses, parabolas, and hyperbolas.
- Recognize, graph, and write equations of parabolas (vertex at origin).
- Recognize, graph, and write equations of ellipses (center at origin).
- Recognize, graph, and write equations of hyperbolas (center at origin).

Why you should learn it

Conics have been used for hundreds of years to model and investigate the paths of comets, planets, and even the paths that airliners follow.

Introduction

Conic sections were discovered during the classical Greek period, 600 to 300 B.C. This early Greek study was largely concerned with the geometric properties of conics. It was not until the early 17th century that the broad applicability of conics became apparent and played a prominent role in the early development of calculus.

A conic section (or simply conic) is the intersection of a plane and a double-napped cone. Notice in Figure 4.18 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane does pass through the vertex, the resulting figure is a degenerate conic, as shown in Figure 4.19.

![Figure 4.18 Basic Conics](image1)

![Figure 4.19 Degenerate Conics](image2)

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \]

However, you will study a third approach, in which each of the conics is defined as a locus (collection) of points satisfying a certain geometric property. For example, in Section 1.1 you saw how the definition of a circle as the collection of all points \((x, y)\) that are equidistant from a fixed point \((h, k)\) led easily to the standard form of the equation of a circle

\[ (x - h)^2 + (y - k)^2 = r^2. \]

Equation of a circle

Recall from Section 1.1 that the center of a circle is at \((h, k)\) and that the radius of the circle is \(r\).
Parabolas

In Section 3.1, you learned that the graph of the quadratic function

\[ f(x) = ax^2 + bx + c \]

is a parabola that opens upward or downward. The following definition of a parabola is more general in the sense that it is independent of the orientation of the parabola.

**Definition of a Parabola**

A parabola is the set of all points \((x, y)\) in a plane that are equidistant from a fixed line, the directrix, and a fixed point, the focus, not on the line. (See Figure 4.20.) The vertex is the midpoint between the focus and the directrix. The axis of the parabola is the line passing through the focus and the vertex.

**Standard Equation of a Parabola (Vertex at Origin)**

The standard form of the equation of a parabola with vertex at \((0, 0)\) and directrix \(y = -p\) is

\[ x^2 = 4py, \quad p \neq 0. \]

Vertical axis

For directrix \(x = -p\), the equation is

\[ y^2 = 4px, \quad p \neq 0. \]

Horizontal axis

The focus is on the axis \(p\) units (directed distance) from the vertex.

For a proof of the standard form of the equation of a parabola, see Proofs in Mathematics on page 380.

Notice that a parabola can have a vertical or a horizontal axis and that a parabola is symmetric with respect to its axis. Examples of each are shown in Figure 4.21.
**Example 1** Finding the Focus of a Parabola

Find the focus of the parabola whose equation is \( y = -2x^2 \).

**Solution**

Because the squared term in the equation involves \( x \), you know that the axis is vertical, and the equation is of the form

\[ x^2 = 4py. \]

You can write the original equation in this form as follows.

\[ x^2 = -\frac{1}{2}y \]

\[ x^2 = 4\left(-\frac{1}{8}\right)y. \]

Write in standard form.

So, \( p = -\frac{1}{8} \). Because \( p \) is negative, the parabola opens downward (see Fig 4.22), and the focus of the parabola is

\[ (0, p) = \left(0, -\frac{1}{8}\right). \]

**Focus**

**CHECKPOINT** Now try Exercise 11.

**Example 2** A Parabola with a Horizontal Axis

Write the standard form of the equation of the parabola with vertex at the origin and focus at \((2, 0)\).

**Solution**

The axis of the parabola is horizontal, passing through \((0, 0)\) and \((2, 0)\), as shown in Figure 4.23. So, the standard form is

\[ y^2 = 4px. \]

Because the focus is \( p = 2 \) units from the vertex, the equation is

\[ y^2 = 4(2)x \]

\[ y^2 = 8x. \]

**CHECKPOINT** Now try Exercise 17.

Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola about its axis. The result is a paraboloid. This has the property that all incoming rays parallel to the axis are reflected through the focus of the paraboloid. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light emanating from the focus of a parabolic reflector used in a flashlight are parallel to one another, as shown in Figure 4.24.
Ellipses

Definition of an Ellipse

An ellipse is the set of all points \((x, y)\) in a plane the sum of whose distances from two distinct fixed points (foci) is constant. See Figure 4.25.

The line through the foci intersects the ellipse at two points (vertices). The chord joining the vertices is the major axis, and its midpoint is the center of the ellipse. The chord perpendicular to the major axis at the center is the minor axis. (See Figure 4.25).

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 4.26. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

The standard form of the equation of an ellipse takes one of two forms, depending on whether the major axis is horizontal or vertical.

Standard Equation of an Ellipse (Center at Origin)

The standard form of the equation of an ellipse centered at the origin with major and minor axes of lengths \(2a\) and \(2b\) (where \(0 < b < c\)) is

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.
\]

The vertices and foci lie on the major axis, \(a\) and \(c\) units, respectively, from the center, as shown in Figure 4.27. Moreover, \(a\), \(b\), and \(c\) are related by the equation \(c^2 = a^2 - b^2\).

Exploration

An ellipse can be drawn using two thumbtacks placed at the foci of the ellipse, a string of fixed length (greater than the distance between the tacks), and a pencil, as shown in Figure 4.26. Try doing this. Vary the length of the string and the distance between the thumbtacks. Explain how to obtain ellipses that are almost circular. Explain how to obtain ellipses that are long and narrow.

In Figure 4.27(a), note that because the sum of the distances from a point on the ellipse to the two foci is constant, it follows that

\[
\text{(Sum of distances from } (0, b) \text{ to foci)} = \text{(sum of distances from } (a, 0) \text{ to foci)}
\]

\[
2\sqrt{b^2 + c^2} = (a + c) + (a - c)
\]

\[
\sqrt{b^2 + c^2} = a.
\]

\[
c^2 = a^2 - b^2.
\]
Example 3  Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse shown in Figure 4.28.

Solution

From Figure 4.28, the foci occur at (−2, 0) and (2, 0). So, the center of the ellipse is (0, 0), the major axis is horizontal, and the ellipse has an equation of the form

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \]

Standard form

Also from Figure 4.28, the length of the major axis is 2a = 6. This implies that a = 3. Moreover, the distance from the center to either focus is c = 2. Finally,

\[ b^2 = a^2 - c^2 = 3^2 - 2^2 = 9 - 4 = 5. \]

Substituting \(a^2 = 3^2\) and \(b^2 = (\sqrt{5})^2\) yields the following equation in standard form.

\[ \frac{x^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1 \]

This equation simplifies to

\[ \frac{x^2}{9} + \frac{y^2}{5} = 1. \]

CHECKPOINT  Now try Exercise 49.

Example 4  Sketching an Ellipse

Sketch the ellipse given by \(4x^2 + y^2 = 36\), and identify the vertices.

Solution

\[ 4x^2 + y^2 = 36 \]  Write original equation.

\[ \frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36} \]  Divide each side by 36.

\[ \frac{x^2}{9} + \frac{y^2}{36} = 1 \]  Write in standard form.

\[ \frac{x^2}{9} + \frac{y^2}{36} = 1 \]  Simplify.

Because the denominator of the \(y^2\)-term is larger than the denominator of the \(x^2\)-term, you can conclude that the major axis is vertical. Moreover, because \(a = 6\), the vertices are (0, −6) and (0, 6). Finally, because \(b = 3\), the endpoints of the minor axis (or co-vertices) are (−3, 0) and (3, 0), as shown in Figure 4.29. Note that you can sketch the ellipse by locating the endpoints of the two axes. Because \(3^2\) is the denominator of the \(x^2\)-term, move three units to the right and left of the center to locate the endpoints of the horizontal axis. Similarly, because \(6^2\) is the denominator of the \(y^2\)-term, move six units up and down from the center to locate the endpoints of the vertical axis.

CHECKPOINT  Now try Exercise 41.
Hyperbolas

The definition of a hyperbola is similar to that of an ellipse. The difference is that for an ellipse the sum of the distances between the foci and a point on the ellipse is constant, whereas for a hyperbola the difference of the distances between the foci and a point on the hyperbola is constant.

**Definition of a Hyperbola**

A hyperbola is the set of all points \((x, y)\) in a plane the difference of whose distances from two distinct fixed points (foci) is a positive constant. See Figure 4.30(a).

The graph of a hyperbola has two disconnected parts (branches). The line through the two foci intersects the hyperbola at two points (vertices). The line segment connecting the vertices is the transverse axis, and the midpoint of the transverse axis is the center of the hyperbola. See Figure 4.30(b).

**Standard Equation of a Hyperbola (Center at Origin)**

The standard form of the equation of a hyperbola with center at the origin (where \(a \neq 0\) and \(b \neq 0\)) is

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}
\]

or

\[
\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \quad \text{Transverse axis is vertical.}
\]

The vertices and foci are, respectively, \(a\) and \(c\) units from the center. Moreover, \(a\), \(b\), and \(c\) are related by the equation \(b^2 = c^2 - a^2\). See Figure 4.31.

**STUDY TIP**

When finding the foci of ellipses and hyperbolas, notice that the relationships between \(a\), \(b\), and \(c\) differ slightly.

Finding the foci of an ellipse:

\[
c^2 = a^2 - b^2
\]

Finding the foci of a hyperbola:

\[
c^2 = a^2 + b^2
\]

**FIGURE 4.31**
Example 5  Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci at \((-3,0)\) and \((3,0)\) and vertices at \((-2,0)\) and \((2,0)\), as shown in Figure 4.32.

Solution

From the graph, you can determine that \(c = 3\), because the foci are three units from the center. Moreover, \(a = 2\) because the vertices are two units from the center. So, it follows that

\[
b^2 = c^2 - a^2 \\
= 3^2 - 2^2 \\
= 9 - 4 \\
= 5.
\]

Because the transverse axis is horizontal, the standard form of the equation is

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.
\]

Finally, substitute \(a^2 = 2^2\) and \(b^2 = (\sqrt{5})^2\) to obtain

\[
\frac{x^2}{2^2} - \frac{y^2}{(\sqrt{5})^2} = 1 \quad \text{Write in standard form.}
\]

\[
\frac{x^2}{4} - \frac{y^2}{5} = 1 \quad \text{Simplify.}
\]

CHECKPOINT  Now try Exercise 69.

An important aid in sketching the graph of a hyperbola is the determination of its asymptotes, as shown in Figure 4.33. Each hyperbola has two asymptotes that intersect at the center of the hyperbola. Furthermore, the asymptotes pass through the corners of a rectangle of dimensions \(2a\) by \(2b\). The line segment of length \(2b\) joining \((0, b)\) and \((0, -b)\) [or \((-b, 0)\) and \((b, 0)\)] is the conjugate axis of the hyperbola.

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

Asymptote: \(y = \frac{b}{a}x\)

\[
\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1
\]

Asymptote: \(y = \frac{a}{b}x\)

(a) Transverse axis is horizontal; conjugate axis is vertical.

(b) Transverse axis is vertical; conjugate axis is horizontal.

FIGURE 4.33
Asymptotes of a Hyperbola (Center at Origin)
The asymptotes of a hyperbola with center at \((0, 0)\) are
\[
y = \frac{b}{a} x \quad \text{and} \quad y = -\frac{b}{a} x.
\]
Transverse axis is horizontal.
or
\[
y = \frac{a}{b} x \quad \text{and} \quad y = -\frac{a}{b} x.
\]
Transverse axis is vertical.

Example 6  Sketching a Hyperbola

Sketch the hyperbola whose equation is
\[
4x^2 - y^2 = 16.
\]

Solution
\[
4x^2 - y^2 = 16 \quad \text{Write original equation.}
\]
\[
\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16} \quad \text{Divide each side by 16.}
\]
\[
\frac{x^2}{4} - \frac{y^2}{16} = 1 \quad \text{Write in standard form.}
\]
\[
\frac{x^2}{4} - \frac{y^2}{16} = 1 \quad \text{Simplify.}
\]

Because the \(x^2\)-term is positive, you can conclude that the transverse axis is horizontal and the vertices occur at \((-2, 0)\) and \((2, 0)\). Moreover, the endpoints of the conjugate axis occur at \((0, -4)\) and \((0, 4)\), and you can sketch the rectangle shown in Figure 4.34. Finally, by drawing the asymptotes through the corners of this rectangle, you can complete the sketch shown in Figure 4.35. Note that the asymptotes are \(y = 2x\) and \(y = -2x\).

\[\text{FIGURE 4.34}\]
\[\text{FIGURE 4.35}\]

CHECKPOINT Now try Exercise 67.
Example 7  Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola that has vertices at \((0, -3)\) and \((0, 3)\) and asymptotes \(y = -2x\) and \(y = 2x\), as shown in Figure 4.36.

Solution

Because the transverse axis is vertical, the asymptotes are of the forms

\[
y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x.
\]

Using the fact that \(y = 2x\) and \(y = -2x\), you can determine that

\[
\frac{a}{b} = 2.
\]

Because \(a = 3\), you can determine that \(b = \frac{3}{2}\). Finally, you can conclude that the hyperbola has the following equation.

\[
\frac{y^2}{9} - \frac{x^2}{\frac{9}{4}} = 1
\]

\(\checkmark\)CHECKPOINT  Now try Exercise 71.

4.3  Exercises

VOCABULARY CHECK: Fill in the blanks.

1. A _______ is the intersection of a plane and a double-napped cone.
2. The equation \((x - h)^2 + (y - k)^2 = r^2\) is the standard form of the equation of a _______ with center \((h, k)\) and radius \(r\).
3. A _______ is the set of all points \((x, y)\) in a plane that are equidistant from a fixed line, called the _______, and a fixed point, called the _______, not on the line.
4. The _______ of a parabola is the midpoint between the focus and the directrix.
5. The line that passes through the focus and the vertex of a parabola is called the _______ of the parabola.
6. An _______ is the set of all points \((x, y)\) in a plane, the sum of whose distances from two distinct fixed points, called _______, is constant.
7. The chord joining the vertices of an ellipse is called the _______ _______, and its midpoint is the _______ of the ellipse.
8. The chord perpendicular to the major axis at the center of an ellipse is called the _______ _______ of the ellipse.
9. A _______ is the set of all points \((x, y)\) in a plane, the difference of whose distances from two distinct fixed points, called _______, is a positive constant.
10. The line segment connecting the vertices of a hyperbola is called the _______ _______, and the midpoint of the line segment is the _______ of the hyperbola.

In Exercises 1–10, match the equation with its graph. If the graph of an equation is not shown, write “not shown.” [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

1. \( x^2 = 2y \)  
2. \( x^2 = -2y \)  
3. \( y^2 = 2x \)  
4. \( y^2 = -2x \)  
5. \( 9y^2 + x^2 = 9 \)  
6. \( x^2 + 9y^2 = 9 \)  
7. \( 9x^2 - y^2 = 9 \)  
8. \( y^2 - 9x^2 = 9 \)  
9. \( x^2 + y^2 = 25 \)  
10. \( x^2 + y^2 = 16 \)

In Exercises 11–16, find the vertex and focus of the parabola and sketch its graph.

11. \( y = \frac{1}{3}x^2 \)  
12. \( y = 2x^2 \)  
13. \( y^2 = -6x \)  
14. \( y^2 = 3x \)  
15. \( x^2 + 8y = 0 \)  
16. \( x + y^2 = 0 \)

In Exercises 17–26, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin.

17. Focus: \((-2, 0)\)  
18. Focus: \((0, -2)\)  
19. Focus: \((0, -\frac{1}{2})\)  
20. Focus: \((\frac{5}{2}, 0)\)  
21. Directrix: \(y = -1\)  
22. Directrix: \(y = 2\)  
23. Directrix: \(x = 3\)  
24. Directrix: \(x = -2\)  
25. Passes through the point \((4, 6)\); horizontal axis  
26. Passes through the point \((-2, -2)\); vertical axis

In Exercises 27–30, find the standard form of the equation of the parabola and determine the coordinates of the focus.

27.  
28.  
29.  
30.  

31. **Flashlight** The light bulb in a flashlight is at the focus of the parabolic reflector, 1.5 centimeters from the vertex of the reflector (see figure). Write an equation for a cross section of the flashlight's reflector with its focus on the positive x-axis and its vertex at the origin.

32. **Satellite Antenna** Write an equation for a cross section of the parabolic television dish antenna shown in the figure.
Model It

33. Suspension Bridge Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway. The cables touch the roadway at the midpoint between the towers.

(a) Draw a sketch of the bridge. Locate the origin of a rectangular coordinate system at the center of the roadway. Label the coordinates of the known points.

(b) Write an equation that models the cables.

(c) Complete the table by finding the height $y$ of the suspension cables over the roadway at a distance of $x$ meters from the center of the bridge.

<table>
<thead>
<tr>
<th>Distance, $x$</th>
<th>0</th>
<th>200</th>
<th>400</th>
<th>500</th>
<th>600</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height, $y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

34. Beam Deflection A simply supported beam (see figure) is 64 feet long and has a load at the center. The deflection of the beam at its center is 1 inch. The shape of the deflected beam is parabolic.

(a) Find an equation of the parabola. (Assume that the origin is at the center of the beam.)

(b) How far from the center of the beam is the deflection $\frac{1}{2}$ inch?

In Exercises 35–42, find the center and vertices of the ellipse and sketch its graph.

35. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

36. $\frac{x^2}{144} + \frac{y^2}{169} = 1$

37. $\frac{x^2}{25/9} + \frac{y^2}{16/9} = 1$

38. $\frac{x^2}{4} + \frac{y^2}{1/4} = 1$

39. $\frac{x^2}{9} + \frac{y^2}{5} = 1$

40. $\frac{x^2}{28} + \frac{y^2}{64} = 1$

41. $4x^2 + y^2 = 1$

42. $4x^2 + 9y^2 = 36$

43. In Exercises 43–52, find the standard form of the equation of the ellipse with the given characteristics and center at the origin.

44.

45.

46.

47. Vertices: $(\pm 5, 0)$; foci: $(\pm 2, 0)$

48. Vertices: $(0, \pm 8)$; foci: $(0, \pm 4)$

49. Foci: $(\pm 5, 0)$; major axis of length 12

50. Foci: $(\pm 2, 0)$; major axis of length 8

51. Vertices: $(0, \pm 5)$; passes through the point $(4, 2)$

52. Major axis vertical; passes through the points $(0, 4)$ and $(2, 0)$

53. Architecture A fireplace arch is to be constructed in the shape of a semicircle. The opening is to have a height of 2 feet at the center and a width of 6 feet along the base (see figure). The contractor draws the outline of the ellipse on the wall by the method shown in Figure 4.26. Give the required positions of the tacks and the length of the string.

54. Architecture A semielliptical arch over a one-way road through a mountain has a major axis of 50 feet and a height at the center of 10 feet.

(a) Draw a rectangular coordinate system on a sketch of the tunnel with the center of the road entering the tunnel at the origin. Identify the coordinates of the known points.

(b) Find an equation of the semielliptical arch over the tunnel.

(c) You are driving a moving truck that has a width of 8 feet and a height of 9 feet. Will the moving truck clear the opening of the arch?
Section 4.3 Conics

77. Art A sculpture has a hyperbolic cross section (see figure).

(a) Write an equation that models the curved sides of the sculpture.

(b) Each unit on the coordinate plane represents 1 foot. Find the width of the sculpture at a height of 5 feet.

78. Optics A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at the focus will be reflected to the other focus. The focus of a hyperbolic mirror (see figure) has coordinates (24, 0). Find the vertex of the mirror if its mount at the top edge of the mirror has coordinates (24, 24).

79. Aeronautics When an airplane travels faster than the speed of sound, the sound waves form a cone behind the airplane. If the airplane is flying parallel to the ground, the sound waves intersect the ground in a hyperbola with the airplane directly above its center (see figure). A sonic boom is heard along the hyperbola. You hear a sonic boom that is audible along a hyperbola with the equation

\[ \frac{x^2}{100} - \frac{y^2}{4} = 1 \]

where \( x \) and \( y \) are measured in miles. What is the shortest horizontal distance you could be to the airplane?
80. **Navigation**  Long distance radio navigation for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci.

Assume that two stations 300 miles apart are positioned on a rectangular coordinate system at points with coordinates (-150, 0) and (150, 0) and that a ship is traveling on a path with coordinates (x, 75), as shown in the figure. Find the x-coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).

![Diagram showing a hyperbola with foci at (-150, 0) and (150, 0) and a ship at (x, 75).]

**Synthesis**

**True or False?** In Exercises 81–83, determine whether the statement is true or false. Justify your answer.

81. The equation \( x^2 - y^2 = 144 \) represents a circle.
82. The major axis of the ellipse \( y^2 + 16x^2 = 64 \) is vertical.
83. It is possible for a parabola to intersect its directrix.

84. **Exploration**  Consider the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a + b = 20.
\]

(a) The area of the ellipse is given by \( A = \pi ab \). Write the area of the ellipse as a function of \( a \).

(b) Find the equation of an ellipse with an area of 264 square centimeters.

(c) Complete the table using your equation from part (a), and make a conjecture about the shape of the ellipse with maximum area.

<table>
<thead>
<tr>
<th>( a )</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Use a graphing utility to graph the area function and use the graph to support your conjecture in part (c).

In Exercises 85–90, identify the conic. Explain your reasoning.

85. \( 4x^2 + 4y^2 - 16 = 0 \)
86. \( 4y^2 - 5x^2 + 20 = 0 \)
87. \( 3y^2 - 6x = 0 \)
88. \( 2x^2 + 4y^2 - 12 = 0 \)
89. \( 4x^2 + y^2 - 16 = 0 \)
90. \( 2x^2 - 12y = 0 \)

91. **Think About It**  How can you tell if an ellipse is a circle from the equation?

92. **Think About It**  Is the graph of \( x^2 + 4y^4 = 4 \) an ellipse? Explain.

93. **Think About It**  The graph of \( x^2 - y^2 = 0 \) is a degenerate conic. Sketch this graph and identify the degenerate conic.

94. **Think About It**  Which part of the graph of the ellipse \( 4x^2 + 9y^2 = 36 \) is represented by each equation? (Do not graph.)

(a) \( x = -\frac{2}{3}\sqrt{9 - y^2} \)
(b) \( y = \frac{2}{3}\sqrt{9 - x^2} \)

95. **Writing**  At the beginning of this section, you learned that each type of conic section can be formed by the intersection of a plane and a double-napped cone. Write a short paragraph describing examples of physical situations in which hyperbolas are formed.

96. **Writing**  Write a paragraph discussing the changes in the shape and orientation of the graph of the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1
\]

as \( a \) increases from 1 to 8.

97. Use the definition of an ellipse to derive the standard form of the equation of an ellipse.

98. Use the definition of a hyperbola to derive the standard form of the equation of a hyperbola.

**Skills Review**

In Exercises 99–102, sketch the graph of the quadratic function without using a graphing utility. Identify the vertex.

99. \( f(x) = x^2 - 8 \)
100. \( f(x) = 25 - x^2 \)
101. \( f(x) = x^2 + 8x + 12 \)
102. \( f(x) = x^2 - 4x - 21 \)

103. Find a polynomial with integer coefficients that has the zeros 3, 2 + i, and 2 - i.

104. Find all the zeros of \( f(x) = 2x^3 - 3x^2 + 50x - 75 \) if one of the zeros is \( x = \frac{5}{2} \).

105. List the possible rational zeros of the function given by \( g(x) = 6x^4 + 7x^2 - 29x^2 - 28x + 20 \).

106. Use a graphing utility to graph the function given by \( h(x) = 2x^4 + x^3 - 19x^2 - 9x + 9 \). Use the graph and the Rational Zero Test to find the zeros of \( h \).
Vertical and Horizontal Shifts of Conics

In Section 4.3 you looked at conic sections whose graphs were in *standard position*. In this section you will study the equations of conic sections that have been shifted vertically or horizontally in the plane.

**Standard Forms of Equations of Conics**

- **Circle**: Center = \((h, k)\); radius = \(r\)
  \[(x - h)^2 + (y - k)^2 = r^2\]

- **Ellipse**: Center = \((h, k)\)
  - Major axis length = \(2a\); minor axis length = \(2b\)

- **Hyperbola**: Center = \((h, k)\)
  - Transverse axis length = \(2a\); conjugate axis length = \(2b\)

- **Parabola**: Vertex = \((h, k)\)
  - Directed distance from vertex to focus = \(p\)
Example 1  Equations of Conic Sections

Identify each conic. Then describe the translation of the graph of the conic.

a. \((x - 1)^2 + (y + 2)^2 = 3^2\)

b. \(\frac{(x - 2)^2}{3^2} + \frac{(y - 1)^2}{2^2} = 1\)

c. \(\frac{(x - 3)^2}{1^2} - \frac{(y - 2)^2}{3^2} = 1\)

d. \((x - 2)^2 = 4(-1)(y - 3)\)

Solution

a. The graph of \((x - 1)^2 + (y + 2)^2 = 3^2\) is a circle whose center is the point \((1, -2)\) and whose radius is 3, as shown in Figure 4.37. The graph of the circle has been shifted one unit to the right and two units downward from standard position.

b. The graph of

\[\frac{(x - 2)^2}{3^2} + \frac{(y - 1)^2}{2^2} = 1\]

is an ellipse whose center is the point \((2, 1)\). The major axis of the ellipse is horizontal and of length \(2\cdot 3 = 6\), and the minor axis of the ellipse is vertical and of length \(2\cdot 2 = 4\), as shown in Figure 4.38. The graph of the ellipse has been shifted two units to the right and one unit upward from standard position.

c. The graph of

\[\frac{(x - 3)^2}{1^2} - \frac{(y - 2)^2}{3^2} = 1\]

is a hyperbola whose center is the point \((3, 2)\). The transverse axis is horizontal and of length \(2\cdot 1 = 2\), and the conjugate axis is vertical and of length \(2\cdot 3 = 6\), as shown in Figure 4.39. The graph of the hyperbola has been shifted three units to the right and two units upward from standard position.

d. The graph of

\((x - 2)^2 = 4(-1)(y - 3)\)

is a parabola whose vertex is the point \((2, 3)\). The axis of the parabola is vertical. The focus is one unit above or below the vertex. Moreover, because \(p = -1\), it follows that the focus lies below the vertex, as shown in Figure 4.40. The graph of the parabola has been reflected in the \(x\)-axis, shifted two units to the left, and shifted three units upward from standard position.

\(\checkmark\) CHECKPOINT  Now try Exercise 1.
Equations of Conics in Standard Form

Example 2  Finding the Standard Form of a Parabola

Find the vertex and focus of the parabola \(x^2 - 2x + 4y - 3 = 0\).

Solution

Complete the square to write the equation in standard form.

\[
\begin{align*}
(x - 1)^2 &= 4(-1)(y - 1) \\
(x - h)^2 &= 4p(y - k) \\
(x - 1)^2 &= 4(-1)(y - 1)
\end{align*}
\]

From this standard form, it follows that \(h = 1\), \(k = 1\), and \(p = -1\). Because the axis is vertical and \(p\) is negative, the parabola opens downward. The vertex is \((h, k) = (1, 1)\) and the focus is \((h, k + p) = (1, 0)\). (See Figure 4.41.)

Checkpoint  Now try Exercise 23.

Example 3  Sketching an Ellipse

Sketch the ellipse \(x^2 + 4y^2 + 6x - 8y + 9 = 0\).

Solution

Complete the square to write the equation in standard form.

\[
\begin{align*}
(x + 3)^2 + 4(y - 1)^2 &= 4 \\
\frac{(x + 3)^2}{4} + \frac{(y - 1)^2}{1} &= 1
\end{align*}
\]

From this standard form, it follows that the center is \((h, k) = (-3, 1)\). Because the denominator of the \(x\)-term is \(a^2 = 4\), the endpoints of the major axis lie two units to the right and left of the center. Similarly, because the denominator of the \(y\)-term is \(b^2 = 1\), the endpoints of the minor axis lie one unit up and down from the center. The ellipse is shown in Figure 4.42.

Checkpoint  Now try Exercise 39.
Example 4  Sketching a Hyperbola

Sketch the hyperbola
\[ y^2 - 4x^2 + 4y + 24x - 41 = 0. \]

Solution

Complete the square to write the equation in standard form.
\[
\begin{align*}
\quad 
\text{Original equation:} &\quad y^2 - 4x^2 + 4y + 24x - 41 = 0 \\
\text{Group terms:} &\quad \left( y^2 + 4y + \ldots \right) - \left( 4x^2 - 24x + \ldots \right) = 41 \\
\text{Factor 4 out of x-terms:} &\quad \left( y^2 + 4y + \frac{4}{4} \right) - 4\left( x^2 - 6x + \frac{9}{4} \right) = 41 + \frac{4}{4} - \frac{4\cdot 9}{4} \\
\text{Add 4 and subtract 4(9) = 36.} &\quad (y + 2)^2 - 4(x - 3)^2 = 9 \\
\text{Write in completed square form.} &\quad \frac{(y + 2)^2}{9} - \frac{4(x - 3)^2}{9} = 1 \\
\text{Divide each side by 9.} &\quad \frac{(y + 2)^2}{9} - \frac{(x - 3)^2}{9/4} = 1 \\
\text{Change 4 to \frac{1}{4}.} &\quad \frac{(y + 2)^2}{3^2} - \frac{(x - 3)^2}{(3/2)^2} = 1 \\
\end{align*}
\]

From this standard form, it follows that the transverse axis is vertical and the center lies at \((h, k) = (3, -2)\). Because the denominator of the \(y\)-term is \(a^2 = 3^2\), you know that the vertices occur three units above and below the center.

\[ (3, -5) \quad \text{and} \quad (3, 1) \quad \text{Vertices} \]

To sketch the hyperbola, draw a rectangle whose top and bottom pass through the vertices. Because the denominator of the \(x\)-term is \(b^2 = (\frac{3}{2})^2\), locate the sides of the rectangle \(\frac{3}{2}\) units to the right and left of the center, as shown in Figure 4.43. Finally, sketch the asymptotes by drawing lines through the opposite corners of the rectangle. Using these asymptotes, you can complete the graph of the hyperbola, as shown in Figure 4.43.

\[ \text{CHECKPOINT} \quad \text{Now try Exercise 63.} \]

To find the foci in Example 4, first find \(c\).
\[
c^2 = a^2 + b^2 \\
\Rightarrow c = \sqrt{a^2 + b^2} = \sqrt{\frac{45}{4}} \Rightarrow c = \frac{3\sqrt{5}}{2}
\]

Because the transverse axis is vertical, the foci lie \(c\) units above and below the center.

\[ (3, -2 + \frac{3\sqrt{5}}{2}) \quad \text{and} \quad (3, -2 - \frac{3\sqrt{5}}{2}) \quad \text{Foci} \]
Example 5  Writing the Equation of an Ellipse

Write the standard form of the equation of the ellipse whose vertices are \((2, -2)\) and \((2, 4)\). The length of the minor axis of the ellipse is 4, as shown in Figure 4.44.

Solution
The center of the ellipse lies at the midpoint of its vertices. So, the center is

\[(h, k) = (2, 1).\]

Because the vertices lie on a vertical line and are six units apart, it follows that the major axis is vertical and has a length of \(2a = 6\). So, \(a = 3\). Moreover, because the minor axis has a length of 4, it follows that \(2b = 4\), which implies that \(b = 2\). Therefore, the standard form of the ellipse is as follows.

\[
\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \quad \text{Major axis is vertical.}
\]

\[
\frac{(x - 2)^2}{2^2} + \frac{(y - 1)^2}{3^2} = 1 \quad \text{Write in standard form.}
\]

Now try Exercise 43.

An interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. For example, Halley’s comet has an elliptical orbit, and reappearance of this comet can be predicted every 76 years. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure 4.45.

If \(p\) is the distance between the vertex and the focus (in meters), and \(v\) is the speed of the comet at the vertex (in meters per second), then the type of orbit is determined as follows.

1. Ellipse: \(v < \sqrt{\frac{2GM}{p}}\)
2. Parabola: \(v = \sqrt{\frac{2GM}{p}}\)
3. Hyperbola: \(v > \sqrt{\frac{2GM}{p}}\)

In each of these relations, \(M = 1.989 \times 10^{30}\) kilograms (the mass of the sun) and \(G = 6.67 \times 10^{-11}\) cubic meter per kilogram-second squared (the universal gravitational constant).

Writing About Mathematics

Identifying Equations of Conics  Use the Internet to research information about the orbits of comets in our solar system. What can you find about the orbits of comets that have been identified since 1970? Write a summary of your results. Identify your source. Does it seem reliable?
4.4 Exercises

**VOCABULARY CHECK:** Match the description of the conic with its standard equation. The equations are labeled (a), (b), (c), (d), (e), (f), and (g).

(a) \( \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \)  
(b) \( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \)  
(c) \( \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \)  
(d) \( \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \)  
(e) \( (x-h)^2 = 4p(y-k) \)  
(f) \( (y-k)^2 = 4p(x-h) \)  
(g) \( (x-h)^2 + (y-k)^2 = r^2 \)

1. Circle  
2. Ellipse with vertical major axis  
3. Parabola with vertical axis  
4. Hyperbola with horizontal transverse axis  
5. Ellipse with horizontal major axis  
6. Parabola with horizontal axis  
7. Hyperbola with vertical transverse axis

**PREREQUISITE SKILLS REVIEW:** Practice and review algebra skills needed for this section at [www.Eduspace.com](http://www.Eduspace.com).

In Exercises 1–6, describe the translation of the graph of the conic.

1. \( (x+2)^2 + (y-1)^2 = 4 \)  
2. \( (y-1)^2 = 4(2)(x+2) \)  
3. \( \frac{(y+3)^2}{4} - (x-1)^2 = 1 \)  
4. \( \frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1 \)  
5. \( \frac{(x+4)^2}{9} + \frac{(y+2)^2}{16} = 1 \)  
6. \( \frac{(x+2)^2}{4} - \frac{(y-3)^2}{9} = 1 \)

In Exercises 7–12, identify the center and radius of the circle.

7. \( x^2 + y^2 = 49 \)  
8. \( x^2 + y^2 = 1 \)  
9. \( (x+3)^2 + (y-8)^2 = 16 \)  
10. \( (x+9)^2 + (y+1)^2 = 36 \)  
11. \( (x-1)^2 + y^2 = 10 \)  
12. \( x^2 + (y+12)^2 = 24 \)

In Exercises 13–16, write the equation of the circle in standard form, and then identify its center and radius.

13. \( x^2 + y^2 - 2x + 6y + 9 = 0 \)  
14. \( x^2 + y^2 - 10x - 6y + 25 = 0 \)  
15. \( 4x^2 + 4y^2 + 12x - 24y + 41 = 0 \)  
16. \( 9x^2 + 9y^2 + 54x - 36y + 17 = 0 \)

In Exercises 17–24, find the vertex, focus, and directrix of the parabola, and sketch its graph.

17. \( (x-1)^2 + 8(y+2) = 0 \)  
18. \( (x+3) + (y-2)^2 = 0 \)  
19. \( (y + \frac{1}{2})^2 = 2(x - 5) \)  
20. \( (x + \frac{1}{2})^2 = 4(y - 3) \)  
21. \( y = \frac{1}{2}(x^2 - 2x + 5) \)  
22. \( 4x - y^2 - 2y - 33 = 0 \)  
23. \( y^2 + 6y + 8x + 25 = 0 \)  
24. \( y^2 - 4y - 4x = 0 \)

In Exercises 25–30, find the standard form of the equation of the parabola with the given characteristics.

25. Vertex: \( (3, 2) \); focus: \( (1, 2) \)  
26. Vertex: \( (-1, 2) \); focus: \( (-1, 0) \)  
27. Vertex: \( (0, 4) \); directrix: \( y = 2 \)  
28. Vertex: \( (-2, 1) \); directrix: \( x = 1 \)  
29. Focus: \( (2, 2) \); directrix: \( x = -2 \)  
30. Focus: \( (0, 0) \); directrix: \( y = 4 \)
32. Satellite Orbit A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour (see figure). If this velocity is multiplied by \( \sqrt{2} \), the satellite will have the minimum velocity necessary to escape Earth’s gravity and it will follow a parabolic path with the center of Earth as the focus.

![Circular orbit diagram](image)

![Parabolic path diagram](image)

33. Projectile Motion A cargo plane is flying at an altitude of 30,000 feet and a speed of 540 miles per hour (792 feet per second). How many feet will a supply crate dropped from the plane travel horizontally before it hits the ground if the path of the crate is modeled by

\[
x^2 = -39,204(y - 30,000)
\]

34. Path of a Projectile The path of a softball is modeled by

\[-12.5(y - 7.125) = (x - 6.25)^2.\]

The coordinates \( x \) and \( y \) are measured in feet, with \( x = 0 \) corresponding to the position from which the ball was thrown.

(a) Use a graphing utility to graph the trajectory of the softball.

(b) Use the trace feature of the graphing utility to approximate the highest point and the range of the trajectory.

35. \[
\frac{(x - 1)^2}{9} + \frac{(y - 5)^2}{25} = 1
\]

36. \[
\frac{(x - 6)^2}{4} + \frac{(y + 7)^2}{16} = 1
\]

37. \[
\frac{(x + 2)^2}{1} + \frac{(y + 1)^2}{4} = 1
\]

38. \[
\frac{(x - 3)^2}{25} + \frac{(y - 8)^2}{9} = 1
\]

39. \[9x^2 + 4y^2 + 36x - 24y + 36 = 0\]

40. \[9x^2 + 4y^2 - 36x + 8y + 31 = 0\]

41. \[16x^2 + 25y^2 - 32x + 50y + 16 = 0\]

42. \[9x^2 + 25y^2 - 36x - 50y + 61 = 0\]

In Exercises 43–52, find the standard form of the equation of the ellipse with the given characteristics.

43. Vertices: \((4, 4), (4, -4)\); minor axis of length 6
44. Vertices: \((-1, 2), (5, 2)\); minor axis of length 4
45. Vertices: \((0, 2), (4, 2)\); minor axis of length 2
46. Foci: \((0, 0), (4, 0)\); major axis of length 8
47. Foci: \((0, 0), (0, 8)\); major axis of length 16
48. Center: \((2, -1);\) vertex: \((2, 3)\);

Minor axis of length 2
49. Center: \((0, 4); a = 2c); vertices: \((-4, 4), (4, 4)\)
50. Center: \((3, 2); a = 3c); foci: \((1, 2), (5, 2)\)
51. Vertices: \((0, 2), (4, 2)\);

Endpoints of the minor axis: \((2, 3), (2, 1)\)
52. Vertices: \((5, 0), (5, 12)\);

Endpoints of the minor axis: \((0, 6), (10, 6)\)
In Exercises 53 and 54, e is called the eccentricity of an ellipse, and is defined by \( e = c/a \). It measures the flatness of the ellipse.

53. Find the standard form of the equation of the ellipse with vertices \((\pm 5, 0)\) and eccentricity \( e = \frac{3}{5} \).

54. Find the standard form of the equation of the ellipse with vertices \((0, \pm 8)\) and eccentricity \( e = \frac{1}{2} \).

55. **Planetary Motion** The planet Pluto moves in an elliptical orbit with the sun at one of the foci, as shown in the figure. The length of half of the major axis, \( a \), is \( 3.67 \times 10^6 \) miles, and the eccentricity is 0.249. Find the smallest distance (perihelion) and the greatest distance (aphelion) of Pluto from the center of the sun.

56. **Australian Football** In Australia, football by Australian Rules (or rugby) is played on elliptical fields. The field can be a maximum of 170 yards wide and a maximum of 200 yards long. Let the center of a field of maximum size be represented by the point \((0, 0)\). Write the standard form of the equation of the ellipse that represents this field.

In Exercises 57–66, find the center, foci, and vertices of the hyperbola, and sketch its graph, using the asymptotes as an aid.

57. \( \frac{(x - 1)^2}{4} - \frac{(y + 2)^2}{1} = 1 \)

58. \( \frac{(x - 1)^2}{144} - \frac{(y - 4)^2}{25} = 1 \)

59. \( (y + 6)^2 - (x - 2)^2 = 1 \)

60. \( \frac{(y - 1)^2}{1/4} - \frac{(x + 3)^2}{1/9} = 1 \)

61. \( 9x^2 - y^2 - 36x - 6y + 18 = 0 \)

62. \( x^2 - 9y^2 + 36y - 72 = 0 \)

63. \( x^2 - 9y^2 + 2x - 54y - 80 = 0 \)

64. \( 16y^2 - 4 - 2x + 64y + 63 = 0 \)

65. \( 9y^2 - 4x^2 + 8x + 18y + 41 = 0 \)

66. \( 11y^2 - 3x^2 + 12x + 44y + 48 = 0 \)

In Exercises 67–76, find the standard form of the equation of the hyperbola with the given characteristics.

67. Vertices: \( (0, 2), (0, 0) \); foci: \( (0, 3), (0, -1) \)

68. Vertices: \( (1, 2), (5, 2) \); foci: \( (0, 2), (6, 2) \)

69. Vertices: \( (2, 0), (6, 0) \); foci: \( (0, 0), (8, 0) \)

70. Vertices: \( (2, 3), (2, -3) \); foci: \( (2, 5), (2, -5) \)

71. Vertices: \( (4, 1), (4, 9) \); foci: \( (4, 0), (4, 10) \)

72. Vertices: \( (-2, 1), (2, 1) \); foci: \( (-3, 1), (3, 1) \)

73. Vertices: \( (2, 3), (2, -3) \); passes through the point \( (0, 5) \)

74. Vertices: \( (-2, 1), (2, 1) \); passes through the point \( (4, 3) \)

75. Vertices: \( (0, 2), (6, 2) \); asymptotes: \( y = \frac{3}{2}x, y = 4 - \frac{1}{2}x \)

76. Vertices: \( (3, 0), (3, 4) \); asymptotes: \( y = \frac{3}{2}x, y = 4 - \frac{1}{2}x \)

In Exercises 77–84, identify the conic by writing its equation in standard form. Then sketch its graph.

77. \( x^2 + y^2 - 6x + 4y + 9 = 0 \)

78. \( x^2 + 4y^2 - 5x + 16y + 21 = 0 \)

79. \( 4x^2 - y^2 - 4x - 3 = 0 \)

80. \( y^2 - 4y - 4x = 0 \)

81. \( 4x^2 + 3y^2 + 8x - 24y + 51 = 0 \)

82. \( 4y^2 - 2x^2 - 4y - 8x - 15 = 0 \)

83. \( 25x^2 - 10x - 200y - 119 = 0 \)

84. \( 4x^2 + 4y^2 - 16y + 15 = 0 \)

**Synthesis**

**True or False?** In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.

85. The conic represented by the equation \( 3x^2 + 2y^2 - 18x - 16y + 58 = 0 \) is an ellipse.

86. The graphs of \( x^2 + 10y - 10x + 5 = 0 \) and \( x^2 + 16y^2 + 10x - 32y - 23 = 0 \) do not intersect.

**Exploration** In Exercises 87 and 88, consider the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

87. Show that the equation of the ellipse can be written as \( \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{c^2(1 - e^2)} = 1 \)

where \( e \) is the eccentricity.

88. Use a graphing utility to graph the ellipse \( \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{c^2(1 - e^2)} = 1 \)

for \( e = 0.95, 0.75, 0.5, 0.25, \) and 0. Make a conjecture about the change in the shape of the ellipse as \( e \) approaches 0.

**Skills Review**

In Exercises 89–92, find the inverse function of \( f \).

89. \( f(x) = 10 - 7x \)

90. \( f(x) = 2 - x^3 \)

91. \( f(x) = \sqrt{x + 8} \)

92. \( f(x) = 3\sqrt{x} + 4 \)