6.2 Right Triangle Trigonometry

The Six Trigonometric Functions

Our first look at the trigonometric functions is from a right triangle perspective. Consider a right triangle, with one acute angle labeled \( \theta \), as shown in Figure 6.23. Relative to the angle \( \theta \), the three sides of the triangle are the hypotenuse, the opposite side (the side opposite the angle \( \theta \)), and the adjacent side (the side adjacent to the angle \( \theta \)).

![Diagram of a right triangle with sides labeled](image)

Using the lengths of these three sides, you can form six ratios that define the six trigonometric functions of the acute angle \( \theta \):

- **Sine** \( \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \)
- **Cosine** \( \cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \)
- **Tangent** \( \tan \theta = \frac{\text{opposite}}{\text{adjacent}} \)
- **Cosecant** \( \csc \theta = \frac{\text{hypotenuse}}{\text{opposite}} \)
- **Secant** \( \sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}} \)
- **Cotangent** \( \cot \theta = \frac{\text{adjacent}}{\text{opposite}} \)

These six functions are normally abbreviated as sin, csc, cos, sec, tan, and cot respectively. In the following definitions it is important to see that \( 0^\circ < \theta < 90^\circ \) (\( \theta \) lies in the first quadrant) and that for such angles the value of each trigonometric function is positive.

Right Triangle Definitions of Trigonometric Functions

Let \( \theta \) be an acute angle of a right triangle. The six trigonometric functions of the angle \( \theta \) are defined as follows. (Note that the functions in the second row are the reciprocals of the corresponding functions in the first row.)

\[
\begin{align*}
\sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
\tan \theta &= \frac{\text{opposite}}{\text{adjacent}} \\
\csc \theta &= \frac{\text{hypotenuse}}{\text{opposite}} \\
\sec \theta &= \frac{\text{hypotenuse}}{\text{adjacent}} \\
\cot \theta &= \frac{\text{adjacent}}{\text{opposite}}
\end{align*}
\]

The abbreviations opp, adj, and hyp represent the lengths of the three sides of a right triangle:

- opp = the length of the side opposite \( \theta \)
- adj = the length of the side adjacent to \( \theta \)
- hyp = the length of the hypotenuse

**Example 1** Evaluating Trigonometric Functions

Use the triangle in Figure 6.24 to find the values of the six trigonometric functions of \( \theta \).

**Solution**

By the Pythagorean Theorem, \((\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2\), it follows that

\[
\text{hyp} = \sqrt{\text{opp}^2 + \text{adj}^2}
\]

\[= \frac{\text{opp}}{\text{adj}} \]

\[= 5\]

So, the six trigonometric functions of \( \theta \) are

\[
\begin{align*}
\sin \theta &= \frac{\text{opp}}{\text{hyp}} = \frac{4}{5} \\
csc \theta &= \frac{\text{hyp}}{\text{opp}} = \frac{5}{4} \\
\cos \theta &= \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} \\
\sec \theta &= \frac{\text{hyp}}{\text{adj}} = \frac{5}{3} \\
\tan \theta &= \frac{\text{opp}}{\text{adj}} = \frac{4}{3} \\
\cot \theta &= \frac{\text{adj}}{\text{opp}} = \frac{3}{4}
\end{align*}
\]

**Example 2** Evaluating Trigonometric Functions of 45°

Find the values of \( \sin 45^\circ \), \( \cos 45^\circ \), and \( \tan 45^\circ \).

**Solution**

Construct a right triangle having 45° as one of its acute angles, as shown in Figure 6.25. Choose the length of the adjacent side to be 1. From geometry, you know that the other acute angle is also 45°. So, the triangle is isosceles and the length of the opposite side is also 1. Using the Pythagorean Theorem, you find the length of the hypotenuse to be \( \sqrt{2} \).

\[
\begin{align*}
\sin 45^\circ &= \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\cos 45^\circ &= \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\
\tan 45^\circ &= \frac{\text{opp}}{\text{adj}} = 1
\end{align*}
\]

**Historical Note**

Georg Joachim Rheticus (1514–1576) was the leading Teutonic mathematical astronomer of the 16th century. He was the first to define the trigonometric functions as ratios of the sides of a right triangle.
Example 3  Evaluating Trigonometric Functions of 30° and 60°

Use the equilateral triangle with sides of length 2 shown in Figure 6.26 to find the values of \(\sin 60°\), \(\cos 60°\), \(\sin 30°\), and \(\cos 30°\).

\[ \text{Figure 6.26} \]

Solution

Use the Pythagorean Theorem and the equilateral triangle in Figure 6.26 to verify the lengths of the sides shown in the figure. For \(\theta = 60°\), you have \(\text{adj} = \sqrt{3}\), and \(\text{hyp} = 2\). So,

\[ \sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2} \]

For \(\theta = 30°\), \(\text{adj} = \frac{\sqrt{3}}{2}\), \(\text{opp} = 1\), and \(\text{hyp} = 2\). So,

\[ \sin 30° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \quad \text{and} \quad \cos 30° = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2} \]

In the box, note that \(\sin 30° = \frac{\sqrt{3}}{2} = \cos 60°\). This occurs because 30° and 60° are complementary angles. In general, it can be shown from the right triangle definitions that cofunctions of complementary angles are equal. That is, if \(\theta\) is an acute angle, the following relationships are true:

\[ \sin(90° - \theta) = \cos \theta \quad \cos(90° - \theta) = \sin \theta \]
\[ \tan(90° - \theta) = \cot \theta \quad \cot(90° - \theta) = \tan \theta \]
\[ \sec(90° - \theta) = \csc \theta \quad \csc(90° - \theta) = \sec \theta \]

Section 6.2  Right Triangle Trigonometry

Trigonometric Identities

In trigonometry, a great deal of time is spent studying relationships (called identities) between trigonometric functions.

Fundamental Trigonometric Identities

**Reciprocal Identities**

\[ \sin \theta = \frac{1}{\csc \theta} \quad \cos \theta = \frac{1}{\sec \theta} \quad \tan \theta = \frac{1}{\cot \theta} \]

**Cotangent Identities**

\[ \csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta} \]

Pythagorean Identities

\[ \sin^2 \theta + \cos^2 \theta = 1 \]
\[ 1 + \tan^2 \theta = \sec^2 \theta \]
\[ 1 + \cot^2 \theta = \csc^2 \theta \]

Note that \(\sin^2 \theta\) represents \((\sin \theta)^2\), \(\cos^2 \theta\) represents \((\cos \theta)^2\), and so on.

Example 4  Applying Trigonometric Identities

Let \(\theta\) be an acute angle such that \(\sin \theta = 0.6\). Find the values of (a) \(\cos \theta\) and (b) \(\tan \theta\) using trigonometric identities.

**Solution**

a. To find the value of \(\cos \theta\), use the Pythagorean identity

\[ \sin^2 \theta + \cos^2 \theta = 1 \]

So, you have

\[ (0.6)^2 + \cos^2 \theta = 1 \]

Subtract \(0.36\) from each side.

\[ \cos^2 \theta = 1 - (0.6)^2 = 0.64 \]

Extract the positive square root.

\[ \cos \theta = \sqrt{0.64} = 0.8 \]

b. Now, knowing the sine and cosine of \(\theta\), you can find the tangent of \(\theta\) to be

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{0.6}{0.8} = 0.75 \]

Use the definitions of \(\cos \theta\) and \(\tan \theta\), and the triangle shown in Figure 6.27, to check these results.

Note: You can use a calculator to convert the answers in Example 3 to decimals. However, the radical form is the exact value and in most cases, the exact value is preferred.
Section 6.2 Right Triangle Trigonometry

Applications Involving Right Triangles

Many applications of trigonometry involve a process called solving right triangles. In this type of application, you are usually given one side of a right triangle and one of the acute angles and asked to find one of the other sides, or you are given two sides and asked to find one of the acute angles.

In Example 7, the angle you are given is the angle of elevation, which represents the angle from the horizontal upward to an object. For objects that lie below the horizontal, it is common to use the term angle of depression, as shown in Figure 6.29.

Example 7 Using Trigonometry to Solve a Right Triangle

A surveyor is standing 115 feet from the base of the Washington Monument, as shown in Figure 6.30. The surveyor measures the angle of elevation to the top of the monument as 78.3°. How tall is the Washington Monument?

Solution

From Figure 6.30, you can see that

\[
\tan 78.3^\circ = \frac{\text{opp}}{\text{adj}} = \frac{z}{x}
\]

where x = 115 and y is the height of the monument. So, the height of the Washington Monument is

\[
y = x \tan 78.3^\circ = 115(4.82882) \approx 555 \text{ feet.}
\]

Example 8 Using Trigonometry to Solve a Right Triangle

An historic lighthouse is 200 yards from a bike path along the edge of a lake. A walkway to the lighthouse is 400 yards long. Find the acute angle \( \theta \) between the bike path and the walkway, as illustrated in Figure 6.31.

Solution

From Figure 6.31, you can see that the sine of the angle \( \theta \) is

\[
\sin \theta = \frac{200}{400} = \frac{1}{2}
\]

Now, you should recognize that \( \theta = 30^\circ \).

Example 5 Applying Trigonometric Identities

Let \( \theta \) be an acute angle such that \( \tan \theta = 3 \). Find the value of each trigonometric function using trigonometric identities:

a. \( \cot \theta \)  

b. \( \sec \theta \)

Solution

a. \( \cot \theta = \frac{1}{\tan \theta} \) 

Reciprocal identity

\[
\cot \theta = \frac{1}{3}
\]

b. \( \sec^2 \theta = 1 + \tan^2 \theta \) 

Pythagorean identity

\[
\sec^2 \theta = 1 + 3^2 = 10
\]

\( \sec \theta = \sqrt{10} \)

Use the definitions of \( \cot \theta \) and \( \sec \theta \), and the triangle shown in Figure 6.28, to check these results.

Remark Now try Exercise 31.

Evaluating Trigonometric Functions with a Calculator

When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired mode of measurement (degrees or radians).

Most calculators do not have keys for the cosecant, secant, and cotangent functions. To evaluate these functions, you can use the \( \csc \) key with their respective reciprocal functions sine, cosine, and tangent. For example, to evaluate \( \csc(\pi/6) \), use the fact that

\[
\csc \frac{\pi}{6} = \frac{1}{\sin(\pi/6)}
\]

and enter the following keystroke sequence in radian mode.

\[
\left[ \frac{\pi}{6} \right] \boxed{\csc} \boxed{\boxed{}} \boxed{=}
\]

Display 2.6133259

Example 6 Using a Calculator

\[
\begin{array}{ccc}
\text{Function} & \text{Mode} & \text{Calculator Keystrokes} & \text{Display} \\
\hline
\sin 76.4^\circ & \text{Degree} & \boxed{76.4} \boxed{=} & 0.9719610 \\
cot 1.5 & \text{Radian} & \left[ \boxed{1.5} \right] \boxed{=} & 0.7079948 \\
\end{array}
\]

Remark Now try Exercise 43.

You could also use the reciprocal identities for sine, cosine, and tangent to evaluate the cosecant, secant, and cotangent functions with a calculator. For instance, you could use the following keystroke sequence to evaluate the function in Example 6(b).

\[
\left[ \boxed{1.5} \right] \boxed{=}
\]

Display 0.7079948
In Example 8, you were able to recognize that $\theta = 30^\circ$ is the acute angle that satisfies the equation $\sin \theta = \frac{1}{2}$. Suppose, however, that you were given the equation $\sin \theta = 0.6$ and were asked to find the acute angle $\theta$. Because

$$\sin 30^\circ = \frac{1}{2}$$

and

$$\sin 45^\circ = \frac{1}{\sqrt{2}}$$

you might guess that $\theta$ lies somewhere between $30^\circ$ and $45^\circ$. In a later section, you will study a method by which a more precise value of $\theta$ can be determined.

**Example 9** Solving a Right Triangle

Find the length $c$ of the skateboard ramp shown in Figure 6.32.

![Figure 6.32](image)

**Solution**

From Figure 6.32, you can see that

$$\sin 18.4^\circ = \frac{opp}{hyp}$$

So, the length of the skateboard ramp is

$$c = \frac{4}{\sin 18.4^\circ}$$

$$\approx 0.3155$$

$$\approx 12.7$$ feet.

**Note:** Now try Exercise 65.

### 6.2 Exercises

**Vocabulary Check:**

1. Match the trigonometric function with its right triangle definition.
   - (a) Sine
   - (b) Cosine
   - (c) Tangent
   - (d) Cosecant
   - (e) Secant
   - (f) Cotangent

   (i) **Hypotenuse**
   (ii) **Adjacent**
   (iii) **Opposite**

   - (g) **Hypotenuse**
   - (h) **Adjacent**
   - (i) **Opposite**

2. In Exercises 2 and 3, fill in the blanks.
   - (a) Label the three sides of a right triangle as the _______ side, the _______ side, and the _______ side.
   - (b) An angle that measures from the horizontal upward to an object is called the angle of _______; whereas an angle that measures from the horizontal downward to an object is called the angle of _______.
   - (c) All the ratios used in trigonometry are the _______ of the hypotenuse and the _______ of the sides.

**Pre-requisite Skills Review:** Practice and review algebra skills needed for this section at www.Eduspace.com.

In Exercises 1–4, find the exact values of the six trigonometric functions of the angle $\theta$ shown in the figure. (Use the Pythagorean Theorem to find the third side and then find the other five trigonometric functions of $\theta$.)

1. $\sin \theta = \frac{3}{5}$
2. $\cos \theta = \frac{4}{5}$
3. $\tan \theta = \frac{3}{4}$
4. $\csc \theta = \frac{5}{3}$

In Exercises 17–26, construct an appropriate triangle to complete the table. ($0 \leq \theta \leq 90^\circ, 0 \leq \theta \leq \pi/2$)

<table>
<thead>
<tr>
<th>Function</th>
<th>$\theta$ (deg)</th>
<th>$\theta$ (rad)</th>
<th>Function Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>30$^\circ$</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>45$^\circ$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>60$^\circ$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\sqrt{3}$</td>
</tr>
<tr>
<td>$\sec \theta$</td>
<td>30$^\circ$</td>
<td>$\frac{\pi}{6}$</td>
<td>2</td>
</tr>
<tr>
<td>$\csc \theta$</td>
<td>45$^\circ$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\sqrt{2}$</td>
</tr>
<tr>
<td>$\cot \theta$</td>
<td>60$^\circ$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
</tr>
</tbody>
</table>

In Exercises 5–8, find the exact values of the six trigonometric functions of the angle $\theta$ for each of the two triangles. Explain why the function values are the same.

<table>
<thead>
<tr>
<th>Triangle</th>
<th>$\theta$ (deg)</th>
<th>$\theta$ (rad)</th>
<th>Function Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>30$^\circ$</td>
<td>$\frac{\pi}{6}$</td>
<td>$\frac{1}{2}$</td>
<td></td>
</tr>
<tr>
<td>45$^\circ$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td></td>
</tr>
<tr>
<td>60$^\circ$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\sqrt{3}$</td>
<td></td>
</tr>
<tr>
<td>90$^\circ$</td>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
In Exercises 27–32, use the given function values, and trigonometric identities (including the cofunction identities), to find the indicated trigonometric functions.

27. \( \sin 60^\circ = \frac{\sqrt{3}}{2} \)
\( \cos 60^\circ = \frac{1}{2} \)
(a) \( \tan 60^\circ \)
(b) \( \sin 30^\circ \)
(c) \( \cos 30^\circ \)
(d) \( \cot 60^\circ \)

28. \( \sin 30^\circ = \frac{1}{2} \)
\( \tan 30^\circ = \frac{\sqrt{3}}{3} \)
(a) \( \cos 30^\circ \)
(b) \( \cot 30^\circ \)
(c) \( \csc 30^\circ \)
(d) \( \sec 30^\circ \)

29. \( \csc \theta = \frac{\sqrt{3}}{3} \)
\( \sec \theta = \sqrt{3} \)
(a) \( \sin \theta \)
(b) \( \cos \theta \)
(c) \( \tan \theta \)
(d) \( \sec(90^\circ - \theta) \)

30. \( \tan \theta = 5 \)
\( \cot \theta = 2 \)
(a) \( \sec \theta \)
(b) \( \csc \theta \)
(c) \( \cos(90^\circ - \theta) \)
(d) \( \sin \theta \)

31. \( \sin \alpha = \frac{1}{2} \)
\( \cos \alpha = \frac{\sqrt{3}}{2} \)
(a) \( \sec \alpha \)
(b) \( \tan \alpha \)
(c) \( \cot \alpha \)
(d) \( \cos(90^\circ - \alpha) \)

32. \( \tan \beta = 5 \)
\( \cot \beta = 0.2 \)
(a) \( \sec \beta \)
(b) \( \csc \beta \)
(c) \( \cos(90^\circ - \beta) \)
(d) \( \sin \beta \)

In Exercises 33–40, use trigonometric identities to transform the left side of the equation into the right side \((0 < \theta < \pi/2)\).

33. \( \sin \theta \cos \theta = \frac{1}{2} \)
34. \( \cos \theta \sec \theta = 1 \)
35. \( \tan \alpha \sec \alpha = \sin \alpha \)
36. \( \cot \alpha \sin \alpha = \cos \alpha \)
37. \( (1 + \cos \theta)(1 - \cos \theta) = \sin^2 \theta \)
38. \( (1 + \sin \theta)(1 - \sin \theta) = \cos^2 \theta \)
39. \( \sin \theta + \tan \theta(\sec \theta - \tan \theta) = 1 \)
40. \( \sin^2 \theta - \cos^2 \theta = 2 \sin^2 \theta - 1 \)
41. \( \sin \theta \cos \theta = \frac{1}{2} \)
42. \( \tan \beta + \cot \beta = \sec \beta \)
\( \tan \beta \)

In Exercises 43–50, use a calculator to evaluate each function. Round your answers to four decimal places. (Be sure the calculator is in the correct angle mode.)

43. \( \tan 22.5^\circ \)
44. \( \sin 16.5^\circ \)
45. \( \cos 16^\circ 18' \)
46. \( \sec 42^\circ 12' \)
47. \( \cot \frac{\pi}{4} \)
48. \( \csc 0.75 \)
49. \( \cos 1 \)
50. \( \sin \left( \frac{\pi}{12} \right) \)

46. Height A six-foot person walks from the base of a broadcasting tower directly toward the tip of the shadow cast by the tower. When the person is 150 feet from the tower and 3 feet from the tip of the shadow, the person's shadow starts to appear beyond the tower's shadow.
(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the tower.
(b) Use a trigonometric function to write an equation involving the unknown quantity.
(c) What is the height of the tower?

51. Angle of Elevation You are sitting on a mountain with a vertical height of 1500 feet. The distance from the top of the mountain to the base is 3000 feet. What is the angle of elevation from the base to the top of the mountain?

52. Width of a River A biologist wants to know the width \( w \) of a river so that she can properly set instruments for studying the pollutants in the water. From point \( A \), the biologist walks downstream 100 feet and sights to point \( C \) (see figure). From this sighting, it is determined that \( \theta = 5^\circ \). How wide is the river?

60. Length A steel cable zip-line is being constructed for a competition in a reality television show. One end of the zip-line is attached to a platform on top of a 150-foot pole. The other end of the zip line is attached to the top of a 5-foot stake. The angle of elevation to the platform is 23° (see figure).
(a) How long is the zip-line?
(b) How far is the stake from the pole?
(c) Contestants take an average of 6 seconds to reach the ground from the top of the zip-line. At what rate are contestants moving down the line? Is this rate consistent with the rate that they maintain while moving down the pole?
6.3 Trigonometric Functions of Any Angle

What you should learn
- Evaluate trigonometric functions of any angle.
- Use reference angles to evaluate trigonometric functions.
- Evaluate trigonometric functions of real numbers.

Why should you learn it
- You can use trigonometric functions to model periodic phenomena such as the growth of a plant or the motion of a planet.

Definitions of Trigonometric Functions of Any Angle

Let \( \theta \) be an angle in standard position with \((x, y)\) a point on the terminal side of \( \theta \) and \( r = \sqrt{x^2 + y^2} \neq 0 \).

- \( \sin \theta = \frac{y}{r} \)
- \( \cos \theta = \frac{x}{r} \)
- \( \tan \theta = \frac{y}{x}, \quad x \neq 0 \)
- \( \cot \theta = \frac{x}{y}, \quad y \neq 0 \)
- \( \sec \theta = \frac{r}{x}, \quad x \neq 0 \)
- \( \csc \theta = \frac{r}{y}, \quad y \neq 0 \)

Because \( r = \sqrt{x^2 + y^2} \) cannot be zero, it follows that the sine and cosine functions are defined for any real value of \( \theta \). However, if \( x = 0 \), the tangent and cotangent of \( \theta \) are undefined. For example, the tangent of 90° is undefined. Similarly, if \( y = 0 \), the cotangent and secant of \( \theta \) are undefined.

Example 1 Evaluating Trigonometric Functions

Let \((-3, 4)\) be a point on the terminal side of \( \theta \). Find the sine, cosine, and tangent of \( \theta \).

Solution

Referring to Figure 6.33, you can see that \( x = -3, y = 4, \) and \( r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5 \).

So, you have the following.

- \( \sin \theta = \frac{4}{5} \)
- \( \cos \theta = \frac{-3}{5} \)
- \( \tan \theta = \frac{4}{3} \)

ENDPOINT Now try Exercise 1.